

# Quantum field theory of fermion mixing

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## Abstract

The fermion mixing transformations are studied in the quantum field theory framework. In particular neutrino mixing is considered and the Fock space of definite flavor states is shown to be unitarily inequivalent to the Fock space of definite mass states. The flavor oscillation formula is computed for two and three flavors mixing and the oscillation amplitude is found to be momentum dependent, a result which may be subject to experimental test. The flavor vacuum state exhibits the structure of  $SU(2)$  generalized coherent state.

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## 1 Introduction

Mixing transformations of fermion fields play a crucial rôle in high energy physics. The original Cabibbo mixing of  $d$  and  $s$  quarks and its extension to the Kobayashi-Maskawa three flavors mixing are essential ingredients in the Standard Model phenomenology [1]. On the other hand, although clear experimental evidence is still missing, it is widely believed that neutrino mixing transformations are the basic tool for further understanding of neutrino phenomenology as well as of solar physics [2].

In contrast with the large body of successful modelling and phenomenological computations (especially for Cabibbo-Kobayashi-Maskawa (CKM) quark mixing, the neutrino mixing still waiting for a conclusive experimental evidence), the quantum field theoretical analysis of the mixing transformations has not been pushed much deeply, as far as we know. The purpose of the present paper is indeed the study of the quantum field theory (QFT) framework of the fermion mixing transformations, thus focusing our attention more on the theoretical structure of fermion mixing than on its phenomenological features.

As we will see, our study is far from being purely academic since, by clarifying the theoretical framework of fermion mixing, we will obtain some results which are also interesting to phenomenology and therefore to the real life of experiments.

In particular, to be definite, we will focus our attention on neutrino mixing transformations and our analysis will lead to some modifications of the neutrino oscillation formulas, which in fact may be subject to experimental test.

The paper is organized as follows. In Section 2 we study the generator of the Pontecorvo neutrino mixing transformations (two flavors mixing for Dirac fields). We show that in the Lehmann-Symanzik-Zimmermann (LSZ) formalism of quantum field theory [3,6,7] the Fock space of the flavor states is unitarily inequivalent to the Fock space of the mass eigenstates in the infinite volume limit. The flavor states are obtained as condensate of massive neutrino pairs and exhibit the structure of  $SU(2)$  coherent states [4]. In Section 3 we exhibit the condensation density as a function of the mixing angle, of the momentum and of the neutrino masses. In Section 4 we derive the neutrino flavor oscillations whose amplitude turns out to be momentum and mass dependent. This is a novel feature with respect to conventional analysis and may be subject to experimental test. In some sense, from the point of view of phenomenology, this is the most interesting result. Nevertheless, the condensate and coherent state structure of the vacuum is by itself a novel and theoretically interesting feature emerging from our analysis. In Section 5 we extend our considerations to three flavors mixing and show how the transformation matrix is obtained in terms of the QFT generators introduced in Section 2. We also obtain the three flavors oscillation formula. Finally, Section 6 is devoted to the conclusions. Although the group theoretical analysis is conceptually simple, specific computations are sometime lengthy and, for the reader convenience, we confine mathematical details to the Appendices.

## 2 The vacuum structure for fermion mixing

For definitiveness we consider the Pontecorvo mixing relations [5], although the following discussion applies to any Dirac fields.

The mixing relations are:

$$\begin{aligned}\nu_e(x) &= \nu_1(x) \cos \theta + \nu_2(x) \sin \theta \\ \nu_\mu(x) &= -\nu_1(x) \sin \theta + \nu_2(x) \cos \theta ,\end{aligned}\tag{2.1}$$

where  $\nu_e(x)$  and  $\nu_\mu(x)$  are the (Dirac) neutrino fields with definite flavors.  $\nu_1(x)$  and  $\nu_2(x)$  are the (free) neutrino fields with definite masses  $m_1$  and  $m_2$ , respectively. The fields  $\nu_1(x)$  and  $\nu_2(x)$  are written as

$$\nu_i(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}, r} [u_{\vec{k}, i}^r \alpha_{\vec{k}, i}^r e^{i\vec{k} \cdot \vec{x}} + v_{\vec{k}, i}^r \beta_{\vec{k}, i}^{r\dagger} e^{-i\vec{k} \cdot \vec{x}}], \quad i = 1, 2 .\tag{2.2}$$

In the following, for simplicity, we will omit the vector notation for  $\vec{k}$  and use the same symbol  $k$  to denote both  $\vec{k}$  and its modulus  $k$ .  $\alpha_{k, i}^r$  and  $\beta_{k, i}^r$ ,  $i = 1, 2$ ,  $r = 1, 2$  are the annihilator operators for the vacuum state  $|0\rangle_{1,2} \equiv |0\rangle_1 \otimes |0\rangle_2$ :  $\alpha_{k, i}^r |0\rangle_{12} = \beta_{k, i}^r |0\rangle_{12} = 0$ . In eq.(2.2) we have included the time dependence in the wave functions. In the following this dependence will be omitted when no misunderstanding arises. The anticommutation relations are:

$$\{\nu_i^\alpha(x), \nu_j^{\beta\dagger}(y)\}_{t=t'} = \delta^3(x - y) \delta_{\alpha\beta} \delta_{ij} , \quad \alpha, \beta = 1, \dots, 4 ,\tag{2.3}$$

and

$$\{\alpha_{k, i}^r, \alpha_{q, j}^{s\dagger}\} = \delta_{kq} \delta_{rs} \delta_{ij}; \quad \{\beta_{k, i}^r, \beta_{q, j}^{s\dagger}\} = \delta_{kq} \delta_{rs} \delta_{ij}, \quad i, j = 1, 2 .\tag{2.4}$$

All other anticommutators are zero. The orthonormality and completeness relations are:

$$\begin{aligned}\sum_{\alpha} u_{k, i}^{r\alpha*} u_{k, i}^{s\alpha} &= \sum_{\alpha} v_{k, i}^{r\alpha*} v_{k, i}^{s\alpha} = \delta_{rs} , \quad \sum_{\alpha} u_{k, i}^{r\alpha*} v_{-k, i}^{s\alpha} = \sum_{\alpha} v_{-k, i}^{r\alpha*} u_{k, i}^{s\alpha} = 0 , \\ \sum_r (u_{k, i}^{r\alpha*} u_{k, i}^{r\beta} + v_{-k, i}^{r\alpha*} v_{-k, i}^{r\beta}) &= \delta_{\alpha\beta} .\end{aligned}\tag{2.5}$$

Eqs.(2.1) (or the ones obtained by inverting them) relate the respective hamiltonians  $H_{1,2}$  (we consider only the mass terms) and  $H_{e,\mu}$  [5]:

$$H_{1,2} = m_1 \nu_1^\dagger \nu_1 + m_2 \nu_2^\dagger \nu_2\tag{2.6}$$

$$H_{e,\mu} = m_{ee} \nu_e^\dagger \nu_e + m_{\mu\mu} \nu_\mu^\dagger \nu_\mu + m_{e\mu} (\nu_e^\dagger \nu_\mu + \nu_\mu^\dagger \nu_e)\tag{2.7}$$

where  $m_{ee} = m_1 \cos^2 \theta + m_2 \sin^2 \theta$ ,  $m_{\mu\mu} = m_1 \sin^2 \theta + m_2 \cos^2 \theta$  and  $m_{e\mu} = (m_2 - m_1) \sin \theta \cos \theta$ .

In QFT the basic dynamics, i.e. the Lagrangian and the resulting field equations, is given in terms of Heisenberg (or interacting) fields. The physical observables are expressed in terms of asymptotic in- (or out-) fields, also called physical or free fields. In the LSZ formalism of QFT [3,6,7], the free fields, say for definitiveness the in-fields, are obtained by the weak limit of the Heisenberg fields for time  $t \rightarrow -\infty$ . The meaning of the weak limit is that the realization of the basic dynamics in terms of the in-fields is not unique so that the limit for  $t \rightarrow -\infty$  (or  $t \rightarrow +\infty$  for the out-fields) is representation dependent. Typical examples are the ones of spontaneously broken symmetry theories, where the same set of Heisenberg field equations describes the normal (symmetric) phase as well as the symmetry broken phase. The representation dependence of the asymptotic limit arises from the existence in QFT of infinitely many unitarily non-equivalent representations of the canonical (anti-)commutation relations [6,7]. Of course, since observables are described in terms of asymptotic fields, unitarily inequivalent representations describe different, i.e. physically inequivalent, phases. It is therefore of crucial importance, in order to get physically meaningful results, to investigate with much care the mapping among Heisenberg or interacting fields and free fields. Such a mapping is usually called the Haag expansion or the dynamical map [6,7]. Only in a very rude and naive approximation we may assume that interacting fields and free fields share the same vacuum state and the same Fock space representation.

We stress that the above remarks apply to QFT, namely to systems with infinite number of degrees of freedom. In quantum mechanics, where finite volume systems are considered, the von Neumann theorem ensures that the representations of the canonical commutation relations are each other unitary equivalent and no problem arises with uniqueness of the asymptotic limit. In QFT, however, the von Neumann theorem does not hold and much more careful attention is required when considering any mapping among interacting and free fields [6,7].

With this warnings, mixing relations such as the relations (2.1) deserve a careful analysis. It is in fact our purpose to investigate the structure of the Fock spaces  $\mathcal{H}_{1,2}$  and  $\mathcal{H}_{e,\mu}$  relative to  $\nu_1(x)$ ,  $\nu_2(x)$  and  $\nu_e(x)$ ,  $\nu_\mu(x)$ , respectively. In particular we want to study the relation among these spaces in the infinite volume limit. We expect that  $\mathcal{H}_{1,2}$  and  $\mathcal{H}_{e,\mu}$  become orthogonal in such a limit, since they represent the Hilbert spaces for free and interacting fields, respectively [6,7]. In the following, as usual, we will perform all computations at finite volume  $V$  and only at the end we will put  $V \rightarrow \infty$ .

Our first step is the study of the generator of eqs.(2.1) and of the underlying group theoretical structure.

Eqs.(2.1) can be put in the form:

$$\begin{aligned}\nu_e^\alpha(x) &= G^{-1}(\theta) \nu_1^\alpha(x) G(\theta) \\ \nu_\mu^\alpha(x) &= G^{-1}(\theta) \nu_2^\alpha(x) G(\theta) ,\end{aligned}\tag{2.8}$$

where  $G(\theta)$  is given by

$$G(\theta) = \exp \left[ \theta \int d^3x \left( \nu_1^\dagger(x) \nu_2(x) - \nu_2^\dagger(x) \nu_1(x) \right) \right] , \quad (2.9)$$

and is (at finite volume) an unitary operator:  $G^{-1}(\theta) = G(-\theta) = G^\dagger(\theta)$ . We indeed observe that, from eqs.(2.8),  $d^2\nu_e^\alpha/d\theta^2 = -\nu_e^\alpha$ ,  $d^2\nu_\mu^\alpha/d\theta^2 = -\nu_\mu^\alpha$ . By using the initial conditions  $\nu_e^\alpha|_{\theta=0} = \nu_1^\alpha$ ,  $d\nu_e^\alpha/d\theta|_{\theta=0} = \nu_2^\alpha$  and  $\nu_\mu^\alpha|_{\theta=0} = \nu_2^\alpha$ ,  $d\nu_\mu^\alpha/d\theta|_{\theta=0} = -\nu_1^\alpha$ , we see that  $G(\theta)$  generates eqs.(2.1).

By introducing the operators

$$S_+ \equiv \int d^3x \nu_1^\dagger(x) \nu_2(x) , \quad S_- \equiv \int d^3x \nu_2^\dagger(x) \nu_1(x) = (S_+)^\dagger , \quad (2.10)$$

$G(\theta)$  can be written as

$$G(\theta) = \exp[\theta(S_+ - S_-)] . \quad (2.11)$$

It is easy to verify that, introducing  $S_3$  and the total charge  $S_0$  as follows

$$S_3 \equiv \frac{1}{2} \int d^3x \left( \nu_1^\dagger(x) \nu_1(x) - \nu_2^\dagger(x) \nu_2(x) \right) , \quad (2.12)$$

$$S_0 \equiv \frac{1}{2} \int d^3x \left( \nu_1^\dagger(x) \nu_1(x) + \nu_2^\dagger(x) \nu_2(x) \right) , \quad (2.13)$$

the  $su(2)$  algebra is closed:

$$[S_+, S_-] = 2S_3 , \quad [S_3, S_\pm] = \pm S_\pm , \quad [S_0, S_3] = [S_0, S_\pm] = 0 . \quad (2.14)$$

Using eq.(2.2) we can expand  $S_+$ ,  $S_-$ ,  $S_3$  and  $S_0$  as follows:

$$S_+ \equiv \sum_k S_+^k = \sum_k \sum_{r,s} (u_{k,1}^{r\dagger} u_{k,2}^s \alpha_{k,1}^{r\dagger} \alpha_{k,2}^s + v_{-k,1}^{r\dagger} u_{k,2}^s \beta_{-k,1}^r \alpha_{k,2}^s + u_{k,1}^{r\dagger} v_{-k,2}^s \alpha_{k,1}^{r\dagger} \beta_{-k,2}^{s\dagger} + v_{-k,1}^{r\dagger} v_{-k,2}^s \beta_{-k,1}^r \beta_{-k,2}^{s\dagger}) , \quad (2.15)$$

$$S_- \equiv \sum_k S_-^k = \sum_k \sum_{r,s} (u_{k,2}^{r\dagger} u_{k,1}^s \alpha_{k,2}^{r\dagger} \alpha_{k,1}^s + v_{-k,2}^{r\dagger} u_{k,1}^s \beta_{-k,2}^r \alpha_{k,1}^s + u_{k,2}^{r\dagger} v_{-k,1}^s \alpha_{k,2}^{r\dagger} \beta_{-k,1}^{s\dagger} + v_{-k,2}^{r\dagger} v_{-k,1}^s \beta_{-k,2}^r \beta_{-k,1}^{s\dagger}) , \quad (2.16)$$

$$S_3 \equiv \sum_k S_3^k = \frac{1}{2} \sum_{k,r} \left( \alpha_{k,1}^{r\dagger} \alpha_{k,1}^r - \beta_{-k,1}^{r\dagger} \beta_{-k,1}^r - \alpha_{k,2}^{r\dagger} \alpha_{k,2}^r + \beta_{-k,2}^{r\dagger} \beta_{-k,2}^r \right) , \quad (2.17)$$

$$S_0 \equiv \sum_k S_0^k = \frac{1}{2} \sum_{k,r} \left( \alpha_{k,1}^{r\dagger} \alpha_{k,1}^r - \beta_{-k,1}^{r\dagger} \beta_{-k,1}^r + \alpha_{k,2}^{r\dagger} \alpha_{k,2}^r - \beta_{-k,2}^{r\dagger} \beta_{-k,2}^r \right) . \quad (2.18)$$

It is interesting to observe that the operatorial structure of eqs.(2.15) and (2.16) is the one of the rotation generator and of the Bogoliubov generator. These structures will be exploited in the following (cf. Section 3 and Appendix D). Using these expansions it is easy to show that the following relations hold:

$$[S_+^k, S_-^k] = 2S_3^k \quad , \quad [S_3^k, S_\pm^k] = \pm S_\pm^k \quad , \quad [S_0^k, S_3^k] = [S_0^k, S_\pm^k] = 0 \quad , \quad (2.19)$$

$$[S_\pm^k, S_\pm^p] = [S_3^k, S_\pm^p] = [S_3^k, S_3^p] = 0 \quad , \quad k \neq p . \quad (2.20)$$

This means that the original  $su(2)$  algebra given in eqs.(2.14) splits into  $k$  disjoint  $su_k(2)$  algebras, given by eqs.(2.19), i.e. we have the group structure  $\bigotimes_k SU_k(2)$ .

To establish the relation between  $\mathcal{H}_{1,2}$  and  $\mathcal{H}_{e,\mu}$  we consider the generic matrix element  ${}_{1,2}\langle a|\nu_1^\alpha(x)|b\rangle_{1,2}$  (a similar argument holds for  $\nu_2^\alpha(x)$ ), where  $|a\rangle_{1,2}$  is the generic element of  $\mathcal{H}_{1,2}$ . Using the inverse of the first of the (2.8), we obtain:

$${}_{1,2}\langle a|G(\theta) \nu_e^\alpha(x) G^{-1}(\theta)|b\rangle_{1,2} = {}_{1,2}\langle a|\nu_1^\alpha(x)|b\rangle_{1,2} . \quad (2.21)$$

Since the operator field  $\nu_e$  is defined on the Hilbert space  $\mathcal{H}_{e,\mu}$ , eq.(2.21) shows that  $G^{-1}(\theta)|a\rangle_{1,2}$  is a vector of  $\mathcal{H}_{e,\mu}$ , so  $G^{-1}(\theta)$  maps  $\mathcal{H}_{1,2}$  to  $\mathcal{H}_{e,\mu}$ :  $G^{-1}(\theta) : \mathcal{H}_{1,2} \mapsto \mathcal{H}_{e,\mu}$ . In particular for the vacuum  $|0\rangle_{1,2}$  we have (at finite volume  $V$ ):

$$|0\rangle_{e,\mu} = G^{-1}(\theta) |0\rangle_{1,2} . \quad (2.22)$$

$|0\rangle_{e,\mu}$  is the vacuum for  $\mathcal{H}_{e,\mu}$ . In fact, from eqs.(2.8) we obtain the positive frequency operators, i.e. the annihilators, relative to the fields  $\nu_e(x)$  and  $\nu_\mu(x)$  as

$$u_{k,e}^{r\alpha} \tilde{\alpha}_{k,e}^r = G^{-1}(\theta) u_{k,1}^{r\alpha} \alpha_{k,1}^r G(\theta) , \quad (2.23a)$$

$$u_{k,\mu}^{r\alpha} \tilde{\alpha}_{k,\mu}^r = G^{-1}(\theta) u_{k,2}^{r\alpha} \alpha_{k,2}^r G(\theta) , \quad (2.23b)$$

$$v_{k,e}^{r\alpha*} \tilde{\beta}_{k,e}^r = G^{-1}(\theta) v_{k,1}^{r\alpha*} \beta_{k,1}^r G(\theta) , \quad (2.23c)$$

$$v_{k,\mu}^{r\alpha*} \tilde{\beta}_{k,\mu}^r = G^{-1}(\theta) v_{k,2}^{r\alpha*} \beta_{k,2}^r G(\theta) . \quad (2.23d)$$

Eqs.(2.23) are obtained by using the linearity of operator  $G(\theta)$ . It is a trivial matter to check that these operators do effectively annihilate  $|0\rangle_{e,\mu}$ .

Furthermore, for the vacuum state  $|0\rangle_{e,\mu}$  the conditions hold:

$$\int d^3x \nu_e^\dagger(x) \nu_e(x) |0\rangle_{e,\mu} = 0 \quad , \quad \int d^3x \nu_\mu^\dagger(x) \nu_\mu(x) |0\rangle_{e,\mu} = 0 , \quad (2.24)$$

as can be verified using eqs.(2.8) and (2.22), or the definitions (2.23).

In Section 3 we will explicitly compute eqs.(2.23), thus giving the dynamical map of the flavor operators in terms of the mass operators.

We observe that  $G^{-1}(\theta) = \exp[\theta(S_- - S_+)]$  is just the generator for generalized coherent states of  $SU(2)$ : the flavor vacuum state is therefore an  $SU(2)$  coherent state. Let us obtain the explicit expression for  $|0\rangle_{e,\mu}$  and investigate the infinite volume limit of eq.(2.22).

Using the Gaussian decomposition,  $G^{-1}(\theta)$  can be written as [4]

$$\exp[\theta(S_- - S_+)] = \exp(-\tan\theta S_+) \exp(-2\ln \cos\theta S_3) \exp(\tan\theta S_-) \quad (2.25)$$

where  $0 \leq \theta < \frac{\pi}{2}$ . Eq.(2.22) then becomes

$$|0\rangle_{e,\mu} = \prod_k \exp(-\tan\theta S_+^k) \exp(-2\ln \cos\theta S_3^k) \exp(\tan\theta S_-^k) |0\rangle_{1,2} . \quad (2.26)$$

The right hand side of eq.(2.26) may be computed by using the relations

$$S_3^k |0\rangle_{1,2} = 0 \quad , \quad S_\pm^k |0\rangle_{1,2} \neq 0 \quad , \quad (S_\pm^k)^2 |0\rangle_{1,2} \neq 0 \quad , \quad (S_\pm^k)^3 |0\rangle_{1,2} = 0 \quad , \quad (2.27)$$

and other useful relations which are given in the Appendix A. The final expression for  $|0\rangle_{e,\mu}$  in terms of  $S_\pm^k$  and  $S_3^k$  is:

$$\begin{aligned} |0\rangle_{e,\mu} = \prod_k |0\rangle_{e,\mu}^k = \prod_k \left[ 1 + \sin\theta \cos\theta (S_-^k - S_+^k) + \frac{1}{2} \sin^2\theta \cos^2\theta ((S_-^k)^2 + (S_+^k)^2) + \right. \\ \left. - \sin^2\theta S_+^k S_-^k + \frac{1}{2} \sin^3\theta \cos\theta (S_-^k (S_+^k)^2 - S_+^k (S_-^k)^2) + \frac{1}{4} \sin^4\theta (S_+^k)^2 (S_-^k)^2 \right] |0\rangle_{1,2} . \end{aligned} \quad (2.28)$$

The state  $|0\rangle_{e,\mu}$  is normalized to 1 (see eq.(2.22)). Eq.(2.28) and eqs.(2.15) and (2.16) exhibit the rich coherent state structure of  $|0\rangle_{e,\mu}$ .

Let us now compute  ${}_{1,2}\langle 0|0\rangle_{e,\mu}$ . We obtain

$${}_{1,2}\langle 0|0\rangle_{e,\mu} = \prod_k \left( 1 - \sin^2\theta {}_{1,2}\langle 0|S_+^k S_-^k|0\rangle_{1,2} + \frac{1}{4} \sin^4\theta {}_{1,2}\langle 0|(S_+^k)^2 (S_-^k)^2|0\rangle_{1,2} \right) \quad (2.29)$$

where (see Appendix B)

$$\begin{aligned} {}_{1,2}\langle 0|S_+^k S_-^k|0\rangle_{1,2} = {}_{1,2}\langle 0| \sum_{\sigma,\tau} \sum_{r,s} (v_{-k,1}^{\sigma\dagger} u_{k,2}^\tau) (u_{k,2}^{s\dagger} v_{-k,1}^r) \beta_{-k,1}^\sigma \alpha_{k,2}^\tau \alpha_{k,2}^{s\dagger} \beta_{-k,1}^{r\dagger} |0\rangle_{1,2} = \\ = \sum_{r,s} |v_{-k,1}^{r\dagger} u_{k,2}^s|^2 \equiv Z_k . \end{aligned} \quad (2.30)$$

In a similar way we find

$${}_{1,2}\langle 0|(S_+^k)^2(S_-^k)^2|0\rangle_{1,2} = Z_k^2. \quad (2.31)$$

Explicitly,  $Z_k$  is given by

$$Z_k = \frac{k^2 [(\omega_{k,2} + m_2) - (\omega_{k,1} + m_1)]^2}{2 \omega_{k,1} \omega_{k,2} (\omega_{k,1} + m_1) (\omega_{k,2} + m_2)} \quad (2.32)$$

where  $\omega_{k,i} = \sqrt{k^2 + m_i^2}$ . The function  $Z_k$  depends on  $k$  only through its modulus and it is always in the interval  $[0, 1]$ . It has a maximum for  $k = \sqrt{m_1 m_2}$  which tends asymptotically to 1 when  $|m_2 - m_1| \rightarrow \infty$ ; also,  $Z_k \rightarrow 0$  when  $k \rightarrow \infty$ .

In conclusion we have

$$\begin{aligned} {}_{1,2}\langle 0|0\rangle_{e,\mu} &= \prod_k \left(1 - \frac{1}{2} \sin^2 \theta Z_k\right)^2 \equiv \prod_k \Gamma(k) = \\ &= \prod_k e^{\ln \Gamma(k)} = e^{\sum_k \ln \Gamma(k)}. \end{aligned} \quad (2.33)$$

From the properties of  $Z_k$  we have that  $\Gamma(k) < 1$  for any value of  $k$  and of the parameters  $m_1$  and  $m_2$ . By using the customary continuous limit relation  $\sum_k \rightarrow \frac{V}{(2\pi)^3} \int d^3k$ , in the infinite volume limit we obtain

$$\lim_{V \rightarrow \infty} {}_{1,2}\langle 0|0\rangle_{e,\mu} = \lim_{V \rightarrow \infty} e^{\frac{V}{(2\pi)^3} \int d^3k \ln \Gamma(k)} = 0 \quad (2.34)$$

Of course, this orthogonality disappears when  $\theta = 0$  and/or when  $m_1 = m_2$  (because in this case  $Z_k = 0$  and no mixing occurs in Pontecorvo theory).

Eq.(2.34) expresses the unitary inequivalence in the infinite volume limit of the flavor and the mass representations and shows the absolutely non-trivial nature of the mixing transformations (2.1). In other words, the mixing transformations induce a physically non-trivial structure in the flavor vacuum which indeed turns out to be an  $SU(2)$  generalized coherent state. In Section 4 we will see how such a vacuum structure may lead to phenomenological consequences in the neutrino oscillations, which possibly may be experimentally tested. From eq.(2.34) we also see that eq.(2.22) is a purely formal expression which only holds at finite volume.

We thus realize the limit of validity of the approximation usually adopted when the mass vacuum state (representation for definite mass operators) is identified with the vacuum for the flavor operators. We point out that even at finite volume the vacua identification is actually an approximation since the flavor vacuum is an  $SU(2)$  generalized coherent state. In such an approximation, the coherent state structure and many physical features are missed.

### 3 The number operator, the dynamical map and the mass sectors



We now calculate the number of the particles condensed in the state  $|0\rangle_{e,\mu}$ .

Let us consider, for example, the  $\alpha_{k,1}$  particles. As usual we define the number operator as  $N_{\alpha_1}^k \equiv \sum_r \alpha_{k,1}^{r\dagger} \alpha_{k,1}^r$  and use the fact that  $N_{\alpha_1}^k$  commutes with  $S_3^p$  and with  $S_{\pm}^p$  for  $p \neq k$ . Then we can write

$${}_{e,\mu}\langle 0|N_{\alpha_1}^k|0\rangle_{e,\mu} = {}_{e,\mu}^k\langle 0|N_{\alpha_1}^k|0\rangle_{e,\mu}^k \quad (3.1)$$

$|0\rangle_{e,\mu}^k$  has been introduced in eq.(2.28). From eq.(3.1) and the relations given in Appendix C we obtain:

$${}_{e,\mu}\langle 0|N_{\alpha_1}^k|0\rangle_{e,\mu} = Z_k \sin^2 \theta . \quad (3.2)$$

The same result is obtained for the number operators  $N_{\alpha_2}^k$ ,  $N_{\beta_1}^k$ ,  $N_{\beta_2}^k$ :

$${}_{e,\mu}\langle 0|N_{\sigma_i}^k|0\rangle_{e,\mu} = Z_k \sin^2 \theta \quad , \quad \sigma = \alpha, \beta \quad , \quad i = 1, 2 . \quad (3.3)$$

Eq.(3.2) gives the condensation density of the flavor vacuum state as a function of the mixing angle  $\theta$ , of the masses  $m_1$  and  $m_2$ , and of the momentum modulus  $k$ . This last feature is particularly interesting since, as we will see, the vacuum acts as a "momentum (or spectrum) analyzer" when time-evolution and flavor oscillations are considered.

We remark that the eq.(3.2) (and (3.3)) clearly shows that the flavor vacuum  $|0\rangle_{e,\mu}$  is not annihilated by the operators  $\alpha_{k,i}^r$ ,  $\beta_{k,i}^r$ ,  $i = 1, 2$ . This is in contrast with the usual treatment where the flavor vacuum  $|0\rangle_{e,\mu}$  is identified with the mass vacuum  $|0\rangle_{1,2}$ .

Notice that, due to the behaviour of  $Z_k$  for high  $k$ , expectation values of  $N_{\sigma_i}^k$  are zero for high  $k$  (the same is true for any operator  $O_i^k$ ,  $i = 1, 2$ , for which is  ${}_{1,2}\langle 0|O_i^k|0\rangle_{1,2} = 0$ ).

In order to explicitly exhibit the dynamical map, eqs.(2.23), it is convenient to redefine the operatorial parts of the fields  $\nu_e(x)$  and  $\nu_\mu(x)$  as (cf. eqs.(2.23))  $u_{k,1}^{r,\alpha} \alpha_{k,e}^r \equiv u_{k,e}^{r,\alpha} \tilde{\alpha}_{k,e}^r$ , etc., so that we can write:

$$\alpha_{k,e}^r \equiv G^{-1}(\theta) \alpha_{k,1}^r G(\theta) , \quad (3.4a)$$

$$\alpha_{k,\mu}^r \equiv G^{-1}(\theta) \alpha_{k,2}^r G(\theta) , \quad (3.4b)$$

$$\beta_{k,e}^r \equiv G^{-1}(\theta) \beta_{k,1}^r G(\theta) , \quad (3.4c)$$

$$\beta_{k,\mu}^r \equiv G^{-1}(\theta) \beta_{k,2}^r G(\theta) . \quad (3.4d)$$

We observe that  $\alpha_{k,l}^r$  and  $\beta_{k,l}^r$ ,  $l = e, \mu$ , depend on time through the time dependence of  $G(\theta)$ . We obtain:

$$\alpha_{k,e}^r = \cos \theta \alpha_{k,1}^r + \sin \theta \sum_s \left( (u_{k,1}^{r\dagger} u_{k,2}^s) \alpha_{k,2}^s + (u_{k,1}^{r\dagger} v_{-k,2}^s) \beta_{-k,2}^{s\dagger} \right) \quad (3.5a)$$

$$\alpha_{k,\mu}^r = \cos \theta \alpha_{k,2}^r - \sin \theta \sum_s \left( (u_{k,2}^{r\dagger} u_{k,1}^s) \alpha_{k,1}^s + (u_{k,2}^{r\dagger} v_{-k,1}^s) \beta_{-k,1}^{s\dagger} \right) \quad (3.5b)$$

$$\beta_{-k,e}^r = \cos \theta \beta_{-k,1}^r + \sin \theta \sum_s \left( (v_{-k,2}^{s\dagger} v_{-k,1}^r) \beta_{-k,2}^s + (u_{k,2}^{s\dagger} v_{-k,1}^r) \alpha_{k,2}^{s\dagger} \right) \quad (3.5c)$$

$$\beta_{-k,\mu}^r = \cos \theta \beta_{-k,2}^r - \sin \theta \sum_s \left( (v_{-k,1}^{s\dagger} v_{-k,2}^r) \beta_{-k,1}^s + (u_{k,1}^{s\dagger} v_{-k,2}^r) \alpha_{k,1}^{s\dagger} \right) \quad (3.5d)$$

Without loss of generality, we can choose the reference frame such that  $k = (0, 0, |k|)$ . This implies that only the products of wave functions with  $r = s$  will survive (see Appendix B). Eqs.(3.5) then assume the simpler form:

$$\alpha_{k,e}^r = \cos \theta \alpha_{k,1}^r + \sin \theta \left( U_k^* \alpha_{k,2}^r + \epsilon^r V_k \beta_{-k,2}^{r\dagger} \right) \quad (3.6a)$$

$$\alpha_{k,\mu}^r = \cos \theta \alpha_{k,2}^r - \sin \theta \left( U_k \alpha_{k,1}^r - \epsilon^r V_k \beta_{-k,1}^{r\dagger} \right) \quad (3.6b)$$

$$\beta_{-k,e}^r = \cos \theta \beta_{-k,1}^r + \sin \theta \left( U_k^* \beta_{-k,2}^r - \epsilon^r V_k \alpha_{k,2}^{r\dagger} \right) \quad (3.6c)$$

$$\beta_{-k,\mu}^r = \cos \theta \beta_{-k,2}^r - \sin \theta \left( U_k \beta_{-k,1}^r + \epsilon^r V_k \alpha_{k,1}^{r\dagger} \right) \quad (3.6d)$$

with  $\epsilon^r = (-1)^r$  and

$$U_k \equiv (u_{k,2}^{r\dagger} u_{k,1}^r) = (v_{-k,1}^{r\dagger} v_{-k,2}^r) \quad (3.7a)$$

$$V_k \equiv \epsilon^r (u_{k,1}^{r\dagger} v_{-k,2}^r) = -\epsilon^r (u_{k,2}^{r\dagger} v_{-k,1}^r) \quad (3.7b)$$

where the time dependence of  $U_k$  and  $V_k$  has been omitted. We have:

$$V_k = |V_k| e^{i(\omega_{k,2} + \omega_{k,1})t} \quad , \quad U_k = |U_k| e^{i(\omega_{k,2} - \omega_{k,1})t} \quad (3.8)$$

$$|U_k| = \left( \frac{\omega_{k,1} + m_1}{2\omega_{k,1}} \right)^{\frac{1}{2}} \left( \frac{\omega_{k,2} + m_2}{2\omega_{k,2}} \right)^{\frac{1}{2}} \left( 1 + \frac{k^2}{(\omega_{k,1} + m_1)(\omega_{k,2} + m_2)} \right) \quad (3.9a)$$

$$|V_k| = \left( \frac{\omega_{k,1} + m_1}{2\omega_{k,1}} \right)^{\frac{1}{2}} \left( \frac{\omega_{k,2} + m_2}{2\omega_{k,2}} \right)^{\frac{1}{2}} \left( \frac{k}{(\omega_{k,2} + m_2)} - \frac{k}{(\omega_{k,1} + m_1)} \right) \quad (3.9b)$$

$$|U_k|^2 + |V_k|^2 = 1 \quad , \quad |V_k|^2 = \frac{1}{2} Z_k \quad (3.10)$$

For notational simplicity in the following we put  $\omega_i \equiv \omega_{k,i}$ .

It is also interesting to exhibit the explicit expression of  $|0\rangle_{e,\mu}^k$  in the reference frame for which  $k = (0, 0, |k|)$  (see Appendix D):

$$\begin{aligned} |0\rangle_{e,\mu}^k = & \prod_r \left[ (1 - \sin^2 \theta |V_k|^2) - \epsilon^r \sin \theta \cos \theta V_k (A^r + B^r) + \right. \\ & \left. + \epsilon^r \sin^2 \theta V_k (U_k^* C^r - U_k D^r) + \sin^2 \theta V_k^2 A^r B^r \right] |0\rangle_{1,2} \end{aligned} \quad (3.11)$$

with

$$A_k^r \equiv \alpha_{k,1}^{r\dagger} \beta_{-k,2}^{r\dagger} \quad , \quad B_k^r \equiv \alpha_{k,2}^{r\dagger} \beta_{-k,1}^{r\dagger} \quad , \quad C_k^r \equiv \alpha_{k,1}^{r\dagger} \beta_{-k,1}^{r\dagger} \quad , \quad D_k^r \equiv \alpha_{k,2}^{r\dagger} \beta_{-k,2}^{r\dagger} \quad (3.12)$$

We observe that eqs.(3.6) can be obtained by a rotation and by a subsequent Bogoliubov transformation. To see this it is convenient to put (cf. eq.(3.10)):

$$|U_k| \equiv \cos \Theta_k \quad , \quad |V_k| \equiv \sin \Theta_k \quad , \quad 0 \leq \Theta_k < \frac{\pi}{4} \quad (3.13)$$

and

$$e^{i(\omega_1 - \omega_2)t} \equiv e^{i\psi} \quad , \quad e^{2i\omega_1 t} \equiv e^{i\phi_1} \quad , \quad e^{2i\omega_2 t} \equiv e^{i\phi_2} \quad (3.14)$$

so that eqs.(3.6) are rewritten as

$$\alpha_{k,e}^r = B_2^{-1} R^{-1} \alpha_{k,1}^r R B_2 \quad (3.15a)$$

$$\beta_{-k,e}^r = B_2^{-1} R^{-1} \beta_{-k,1}^r R B_2 \quad (3.15b)$$

$$\alpha_{k,\mu}^r = B_1^{-1} R^{-1} \alpha_{k,2}^r R B_1 \quad (3.15c)$$

$$\beta_{-k,\mu}^r = B_1^{-1} R^{-1} \beta_{-k,2}^r R B_1 \quad (3.15d)$$

where

$$R = \exp \left\{ \theta \sum_{k,r} \left[ \left( \alpha_{k,1}^{r\dagger} \alpha_{k,2}^r + \beta_{-k,1}^{r\dagger} \beta_{-k,2}^r \right) e^{i\psi} - \left( \alpha_{k,2}^{r\dagger} \alpha_{k,1}^r + \beta_{-k,2}^{r\dagger} \beta_{-k,1}^r \right) e^{-i\psi} \right] \right\} \quad (3.16)$$

$$B_1 = \exp \left\{ - \sum_{k,r} \Theta_k \epsilon^r \left[ \alpha_{k,1}^r \beta_{-k,1}^r e^{-i\phi_1} - \beta_{-k,1}^{r\dagger} \alpha_{k,1}^{r\dagger} e^{i\phi_1} \right] \right\} \quad (3.17)$$

$$B_2 = \exp \left\{ \sum_{k,r} \Theta_k \epsilon^r \left[ \alpha_{k,2}^r \beta_{-k,2}^r e^{-i\phi_2} - \beta_{-k,2}^{r\dagger} \alpha_{k,2}^{r\dagger} e^{i\phi_2} \right] \right\} \quad (3.18)$$

By use of these relations and noting that  $R|0\rangle_{1,2} = |0\rangle_{1,2}$ , we can "separate" the sectors  $\{|0(\Theta)\rangle_1\}$  and  $\{|0(\Theta)\rangle_2\}$  out of the full representation space  $\{|0\rangle_{e,\mu}\}$  :  $\{|0(\Theta)\rangle_1\} \otimes \{|0(\Theta)\rangle_2\} \subset \{|0\rangle_{e,\mu}\}$ .

The states  $|0(\Theta)\rangle_1$  and  $|0(\Theta)\rangle_2$  are respectively obtained as:

$$|0(\Theta)\rangle_1 \equiv B_1^{-1}(\Theta) |0\rangle_1 = \prod_{k,r} \left( \cos \Theta_k + \epsilon^r e^{i\phi_1} \sin \Theta_k \beta_{-k,1}^{r\dagger} \alpha_{k,1}^{r\dagger} \right) |0\rangle_1 \quad (3.19)$$

$$|0(\Theta)\rangle_2 \equiv B_2^{-1}(\Theta) |0\rangle_2 = \prod_{k,r} \left( \cos \Theta_k - \epsilon^r e^{i\phi_2} \sin \Theta_k \beta_{-k,2}^{r\dagger} \alpha_{k,2}^{r\dagger} \right) |0\rangle_2 \quad (3.20)$$

If one wants to work with the "mass" sectors  $\{|0(\Theta)\rangle_1\}$  and  $\{|0(\Theta)\rangle_2\}$ , the tensor product formalism must be used, e.g.

$$\begin{aligned} (O_1 + O_2) (|0(\Theta)\rangle_1 \otimes |0(\Theta)\rangle_2) &\equiv (O_1 \otimes I + I \otimes O_2) (|0(\Theta)\rangle_1 \otimes |0(\Theta)\rangle_2) = \\ &= O_1 |0(\Theta)\rangle_1 \otimes |0(\Theta)\rangle_2 + |0(\Theta)\rangle_1 \otimes O_2 |0(\Theta)\rangle_2 \end{aligned} \quad (3.21)$$

with  $O_i$ ,  $i = 1, 2$ , any product of  $\nu_i$  neutrino field operators. For example, we have  $|0\rangle_{1,2} \equiv |0\rangle_1 \otimes |0\rangle_2$ ,  $\alpha_{k,1}^r \equiv \alpha_{k,1}^r \otimes I$ ,  $\alpha_{k,2}^r \equiv I \otimes \alpha_{k,2}^r$ , so that  $\alpha_{k,1}^r \alpha_{k,2}^{r\dagger} = (\alpha_{k,1}^r \otimes I) (I \otimes \alpha_{k,2}^{r\dagger}) = \alpha_{k,1}^r \otimes \alpha_{k,2}^{r\dagger}$  and

$$R^{-1} \alpha_{k,1}^r R = \cos \theta (\alpha_{k,1}^r \otimes I) + e^{i\psi} \sin \theta (I \otimes \alpha_{k,2}^r) \quad (3.22)$$

$$B_2^{-1} R^{-1} \alpha_{k,1}^r R B_2 = \cos \theta (\alpha_{k,1}^r \otimes I) + e^{i\psi} \sin \theta (I \otimes \alpha_{k,2}^r(\Theta)) \quad (3.23)$$

with

$$\alpha_{k,2}^r(\Theta) = \cos \Theta_k \alpha_{k,2}^r + \epsilon^r e^{i\phi_2} \sin \Theta_k \beta_{-k,2}^{r\dagger} \quad (3.24)$$

and

$$B_2^{-1} |0\rangle_{1,2} = (|0\rangle_1 \otimes |0(\Theta)\rangle_2) . \quad (3.25)$$

We note that  $|0(\Theta)\rangle_i$ ,  $i = 1, 2$ , are the vacuum states for  $\alpha_{k,i}^r(\Theta) = B(\Theta)_i^{-1} \alpha_{k,i}^r B(\Theta)_i$  and  $\beta_{k,i}^r(\Theta) = B(\Theta)_i^{-1} \beta_{k,i}^r B(\Theta)_i$  operators.

We also note that  $\alpha_{k,e}^r (|0\rangle_1 \otimes |0(\Theta)\rangle_2) = \beta_{k,e}^r (|0\rangle_1 \otimes |0(\Theta)\rangle_2) = 0$ ;  $\alpha_{k,\mu}^r (|0(\Theta)\rangle_1 \otimes |0\rangle_2) = \beta_{k,\mu}^r (|0(\Theta)\rangle_1 \otimes |0\rangle_2) = 0$ , but  $\alpha_{k,e}^r (|0(\Theta)\rangle_1 \otimes |0\rangle_2) \neq 0$ ,  $\alpha_{k,\mu}^r (|0\rangle_1 \otimes |0(\Theta)\rangle_2) \neq 0$ , etc.. Moreover,  ${}_2\langle 0(\Theta) | N_{\sigma_2}^{k,r} | 0(\Theta)\rangle_2 = \sin^2 \Theta_k$ ,  ${}_1\langle 0(\Theta) | N_{\sigma_1}^{k,r} | 0(\Theta)\rangle_1 = \sin^2 \Theta_k$ , and  ${}_2\langle 0(\Theta) | N_{\sigma_1}^{k,r} | 0(\Theta)\rangle_2 = {}_1\langle 0(\Theta) | N_{\sigma_2}^{k,r} | 0(\Theta)\rangle_1 = 0$ ,  $\sigma = \alpha, \beta$ , which show the condensate structure of the sectors  $\{|0(\Theta)\rangle_i\}$ ,  $i = 1, 2$ .

Finally, we observe that  $|0(\Theta)\rangle_i$  can be written as

$$|0(\Theta)\rangle_i = \exp\left(-\frac{S_{\alpha_i}}{2}\right) |I_i\rangle = \exp\left(-\frac{S_{\beta_i}}{2}\right) |I_i\rangle \quad (3.26)$$

with  $|I_i\rangle \equiv \exp\left(\sum_{k,r} (-1)^{i+1} \epsilon^r e^{i\phi_i} \beta_{-k,i}^{r\dagger} \alpha_{k,i}^{r\dagger}\right) |0\rangle_{1,2}$ , and

$$S_{\alpha_i} = - \sum_{k,r} \left( \alpha_{k,i}^{r\dagger} \alpha_{k,i}^r \log \sin^2 \Theta_k + \alpha_{k,i}^r \alpha_{k,i}^{r\dagger} \log \cos^2 \Theta_k \right) , \quad i = 1, 2 . \quad (3.27)$$

A similar expression holds for  $S_{\beta_i}$ . It is known that  $S_{\alpha_i}$  (or  $S_{\beta_i}$ ) can be interpreted as the entropy function associated to the vacuum condensate [7].

#### 4 Neutrino oscillations

We are now ready to study the flavor oscillations. In order to compare our result with the conventional one [1,2,5], we first reproduce the usual oscillation formula.

In the original Pontecorvo and collaborators treatment [5], the vacuum state for definite flavor neutrinos is identified with the vacuum state for definite mass neutrinos:  $|0\rangle_{e,\mu} = |0\rangle_{1,2} \equiv |0\rangle$ . As we have shown in the previous Section, such an identification is not possible in QFT; however, it is allowed at finite volume where no problem of unitary inequivalence arises in the choice of the Hilbert space. As already observed, even at finite volume, the vacua identification is only an approximation. For shortness we refer to such an identification simply as to the finite volume approximation, meaning by that the approximation which is allowed at finite volume.

The number operators relative to electronic and muonic neutrinos are

$$\begin{aligned} N_{\alpha_e}^{k,r} &= \alpha_{k,e}^{r\dagger} \alpha_{k,e}^r = \\ &= \cos^2 \theta \alpha_{k,1}^{r\dagger} \alpha_{k,1}^r + \sin^2 \theta \alpha_{k,2}^{r\dagger} \alpha_{k,2}^r + \sin \theta \cos \theta (\alpha_{k,1}^{r\dagger} \alpha_{k,2}^r + \alpha_{k,2}^{r\dagger} \alpha_{k,1}^r) , \end{aligned} \quad (4.1)$$

$$\begin{aligned} N_{\alpha_\mu}^{k,r} &= \alpha_{k,\mu}^{r\dagger} \alpha_{k,\mu}^r = \\ &= \cos^2 \theta \alpha_{k,2}^{r\dagger} \alpha_{k,2}^r + \sin^2 \theta \alpha_{k,1}^{r\dagger} \alpha_{k,1}^r - \sin \theta \cos \theta (\alpha_{k,1}^{r\dagger} \alpha_{k,2}^r + \alpha_{k,2}^{r\dagger} \alpha_{k,1}^r) , \end{aligned} \quad (4.2)$$

and, obviously,

$$\langle 0 | N_{\alpha_e}^{k,r} | 0 \rangle = \langle 0 | N_{\alpha_\mu}^{k,r} | 0 \rangle = 0 . \quad (4.3)$$

The one electronic neutrino state is of the form

$$|\alpha_{k,e}^r\rangle = \cos \theta |\alpha_{k,1}^r\rangle + \sin \theta |\alpha_{k,2}^r\rangle . \quad (4.4)$$

where  $|\alpha_{k,e}^r\rangle \equiv \alpha_{k,e}^{r\dagger} |0\rangle$ ,  $|\alpha_{k,1}^r\rangle \equiv \alpha_{k,1}^{r\dagger} |0\rangle$ ,  $|\alpha_{k,2}^r\rangle \equiv \alpha_{k,2}^{r\dagger} |0\rangle$ . The time evolution of this state is controlled by the time evolution of  $|\alpha_{k,1}^r\rangle$  and  $|\alpha_{k,2}^r\rangle$

$$|\alpha_{k,e}^r(t)\rangle = e^{-iH_{1,2}t} |\alpha_{k,e}^r\rangle = \cos \theta e^{-i\omega_1 t} |\alpha_{k,1}^r\rangle + \sin \theta e^{-i\omega_2 t} |\alpha_{k,2}^r\rangle . \quad (4.5)$$

We have

$$\langle \alpha_{k,e}^r | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r \rangle = 1 \quad (4.6)$$

and

$$\begin{aligned} \langle \alpha_{k,e}^r(t) | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r(t) \rangle &= \cos^4 \theta + \sin^4 \theta + \sin^2 \theta \cos^2 \theta (e^{-i(\omega_2 - \omega_1)t} + e^{+i(\omega_2 - \omega_1)t}) = \\ &= 1 - \sin^2 2\theta \sin^2 \left( \frac{\Delta\omega}{2} t \right) . \end{aligned} \quad (4.7)$$

The number of  $\alpha_e$  particles therefore oscillates in time with a frequency given by the difference  $\Delta\omega$  in the energies of the physical components  $\alpha_1$  and  $\alpha_2$ . This oscillation is a flavor oscillation since we have at the same time:

$$\langle \alpha_{k,e}^r(t) | N_{\alpha_\mu}^{k,r} | \alpha_{k,e}^r(t) \rangle = \sin^2 2\theta \sin^2 \left( \frac{\Delta\omega}{2} t \right) \quad (4.8)$$

so that

$$\langle \alpha_{k,e}^r(t) | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r(t) \rangle + \langle \alpha_{k,e}^r(t) | N_{\alpha_\mu}^{k,r} | \alpha_{k,e}^r(t) \rangle = 1. \quad (4.9)$$

Notice that the traditional derivation of eq.(4.7) is the same as the one presented above since  $\langle \alpha_{k,e}^r(t) | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r(t) \rangle = |\langle \alpha_{k,e}^r | \alpha_{k,e}^r(t) \rangle|^2$ , as can be seen from eqs.(4.4) and (4.5). Eqs.(4.7)-(4.9) are the well known results.

Let us now go to the QFT framework.

We have seen that  $|0\rangle_{1,2}$  and  $|0\rangle_{e,\mu}$  are orthogonal in the infinite volume limit. We choose to work in the physical (mass) representation  $|0\rangle_{1,2}$  (same conclusions are of course reached by working, with due changes, in the flavor representation  $|0\rangle_{e,\mu}$ ) in order to follow the time evolution of the physical components  $|\alpha_{k,1}^r\rangle$  and  $|\alpha_{k,2}^r\rangle$ .

The number operators are now (cf. eqs.(3.6))

$$\begin{aligned} N_{\alpha_e}^{k,r} &= \alpha_{k,e}^{r\dagger} \alpha_{k,e}^r = \\ &= \cos^2 \theta \alpha_{k,1}^{r\dagger} \alpha_{k,1}^r + \sin^2 \theta |U_k|^2 \alpha_{k,2}^{r\dagger} \alpha_{k,2}^r + \sin^2 \theta |V_k|^2 \beta_{k,2}^r \beta_{k,2}^{r\dagger} + \\ &+ \sin \theta \cos \theta \left( U_k^* \alpha_{k,1}^{r\dagger} \alpha_{k,2}^r + U_k \alpha_{k,2}^{r\dagger} \alpha_{k,1}^r + \epsilon^r V_k^* \beta_{k,2}^r \alpha_{k,1}^r + \epsilon^r V_k \alpha_{k,1}^{r\dagger} \beta_{k,2}^{r\dagger} \right) + \\ &+ \epsilon^r \sin^2 \theta \left( V_k U_k \alpha_{k,2}^{r\dagger} \beta_{k,2}^{r\dagger} + V_k^* U_k^* \beta_{k,2}^r \alpha_{k,2}^r \right), \end{aligned} \quad (4.10)$$

$$\begin{aligned} N_{\alpha_\mu}^{k,r} &= \alpha_{k,\mu}^{r\dagger} \alpha_{k,\mu}^r = \\ &= \cos^2 \theta \alpha_{k,2}^{r\dagger} \alpha_{k,2}^r + \sin^2 \theta |U_k|^2 \alpha_{k,1}^{r\dagger} \alpha_{k,1}^r + \sin^2 \theta |V_k|^2 \beta_{k,1}^r \beta_{k,1}^{r\dagger} + \\ &- \sin \theta \cos \theta \left( U_k^* \alpha_{k,1}^{r\dagger} \alpha_{k,2}^r + U_k \alpha_{k,2}^{r\dagger} \alpha_{k,1}^r - \epsilon^r V_k^* \beta_{k,1}^r \alpha_{k,2}^r - \epsilon^r V_k \alpha_{k,2}^{r\dagger} \beta_{k,1}^{r\dagger} \right) + \\ &- \epsilon^r \sin^2 \theta \left( V_k U_k^* \alpha_{k,1}^{r\dagger} \beta_{k,1}^{r\dagger} + V_k^* U_k^* \beta_{k,1}^r \alpha_{k,1}^r \right). \end{aligned} \quad (4.11)$$

The one electronic neutrino state is given by

$$|\alpha_{k,e}^r\rangle \equiv \alpha_{k,e}^{r\dagger} |0\rangle_{1,2} = \cos \theta |\alpha_{k,1}^r\rangle + \sin \theta U_k |\alpha_{k,2}^r\rangle \quad (4.12)$$

with  $|\alpha_{k,i}^r\rangle \equiv \alpha_{k,i}^{r\dagger} |0\rangle_{1,2}$ ,  $i = 1, 2$ . The action of the flavor operator  $\alpha_{k,e}^{r\dagger}$  is defined on  $|0\rangle_{1,2}$  through the mapping (3.6) (see also eqs.(3.4)). Note that, as it must be,

$$\langle \alpha_{k,e}^r | \alpha_{k,e}^r \rangle = 1 - \sin^2 \theta |V_k|^2 = 1 - {}_{1,2}\langle 0 | \alpha_{k,e}^{r\dagger} \alpha_{k,e}^r | 0 \rangle_{1,2}, \quad (4.13)$$

since  $\alpha_{k,e}^r |0\rangle_{1,2} \neq 0$ .

The time evolution of the state  $|\alpha_{k,e}^r\rangle$  is given by

$$|\alpha_{k,e}^r(t)\rangle = e^{-iH_{1,2}t} |\alpha_{k,e}^r\rangle = \cos \theta e^{-i\omega_1 t} |\alpha_{k,1}^r\rangle + \sin \theta U_k e^{-i\omega_2 t} |\alpha_{k,2}^r\rangle. \quad (4.14)$$

We have:

$$\langle \alpha_{k,e}^r | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r \rangle = 1 - \sin^2 \theta |V_k|^2 = 1 - {}_{1,2}\langle 0 | N_{\alpha_e}^{k,r} | 0 \rangle_{1,2} \quad (4.15)$$

so that,

$$\langle \alpha_{k,e}^r | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r \rangle + {}_{1,2}\langle 0 | N_{\alpha_e}^{k,r} | 0 \rangle_{1,2} = 1. \quad (4.16)$$

We note that the expectation value of  $N_{\alpha_e}^{k,r}$  in the vacuum  $|0\rangle_{1,2}$  provides an essential contribution to the normalization equation (4.16) (see also eq.(4.13)).

It is also interesting to observe that the term  ${}_{1,2}\langle 0 | N_{\alpha_e}^{k,r} | 0 \rangle_{1,2}$  plays the rôle of zero point contribution when considering the energy contribution of  $\alpha_e^{k,r}$  particles.

Note that we also have:

$${}_{1,2}\langle 0 | N_{\sigma_l}^{k,r} | 0 \rangle_{1,2} = \sin^2 \theta |V_k|^2, \quad \sigma = \alpha, \beta, \quad l = e, \mu, \quad (4.17)$$

and

$$\langle \alpha_{k,e}^r | N_{\alpha_\mu}^{k,r} | \alpha_{k,e}^r \rangle = \sin^2 \theta |V_k|^2 \left( 1 - \sin^2 \theta |V_k|^2 \right). \quad (4.18)$$

Eqs.(4.17) show the condensate structure of  $|0\rangle_{1,2}$  in terms of definite flavor fields and of course are analogous to eqs.(3.3) which show the condensate structure of  $|0\rangle_{e,\mu}$  in terms of definite mass fields.

We also have

$$\begin{aligned} & \langle \alpha_{k,e}^r(t) | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r(t) \rangle = \\ & = \cos^4 \theta + |U_k|^2 \sin^4 \theta + |V_k|^2 \sin^2 \theta \cos^2 \theta + |U_k|^2 \sin^2 \theta \cos^2 \theta (e^{-i(\omega_2 - \omega_1)t} + e^{+i(\omega_2 - \omega_1)t}) = \\ & = \left( 1 - \sin^2 \theta |V_k|^2 \right) - |U_k|^2 \sin^2 2\theta \sin^2 \left( \frac{\Delta\omega}{2} t \right). \end{aligned} \quad (4.19)$$

This result reproduces the one obtained in the finite volume approximation (cf. eq.(4.7)) when  $|U_k| \rightarrow 1$  (and  $|V_k| \rightarrow 0$ ). The fraction of  $\alpha_\mu^{k,r}$  particles in the same state is

$$\begin{aligned} & \langle \alpha_{k,e}^r(t) | N_{\alpha_\mu}^{k,r} | \alpha_{k,e}^r(t) \rangle = \\ & = |U_k|^2 \sin^2 2\theta \sin^2 \left( \frac{\Delta\omega}{2} t \right) + \sin^2 \theta |V_k|^2 \left( 1 - \sin^2 \theta |V_k|^2 \right), \end{aligned} \quad (4.20)$$

where we recognize the contribution from the  $\alpha_{k,\mu}^r$  condensate in the state  $|\alpha_{k,e}^r\rangle$  (cf. eq.(4.18)). Eq.(4.19) is to be compared with the approximated one (4.7). Note that

$$\langle \alpha_{k,e}^r(t) | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r(t) \rangle + \langle \alpha_{k,e}^r(t) | N_{\alpha_\mu}^{k,r} | \alpha_{k,e}^r(t) \rangle = \langle \alpha_{k,e}^r | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r \rangle + \langle \alpha_{k,e}^r | N_{\alpha_\mu}^{k,r} | \alpha_{k,e}^r \rangle. \quad (4.21)$$

The normalization relation is written as

$$\langle \alpha_{k,e}^r(t) | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r(t) \rangle + {}_{1,2}\langle 0 | N_{\alpha_e}^{k,r} | 0 \rangle_{1,2} + \langle \alpha_{k,e}^r(t) | N_{\alpha_\mu}^{k,r} | \alpha_{k,e}^r(t) \rangle - \langle \alpha_{k,e}^r | N_{\alpha_\mu}^{k,r} | \alpha_{k,e}^r \rangle = 1. \quad (4.22)$$

We may also use  $\mathcal{N}_k \equiv \langle \alpha_{k,e}^r | \alpha_{k,e}^r \rangle$  as normalization factor and write eqs.(4.15), (4.18), (4.19), (4.20) and the normalization relation as

$$\frac{1}{\mathcal{N}_k} \langle \alpha_{k,e}^r | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r \rangle = 1 \quad (4.23)$$

$$\frac{1}{\mathcal{N}_k} \langle \alpha_{k,e}^r | N_{\alpha_\mu}^{k,r} | \alpha_{k,e}^r \rangle = \sin^2 \theta |V_k|^2 \quad (4.24)$$

$$\frac{1}{\mathcal{N}_k} \langle \alpha_{k,e}^r(t) | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r(t) \rangle = 1 - R_k \sin^2 2\theta \sin^2 \left( \frac{\Delta\omega}{2} t \right) \quad (4.25)$$

$$\frac{1}{\mathcal{N}_k} \langle \alpha_{k,e}^r(t) | N_{\alpha_\mu}^{k,r} | \alpha_{k,e}^r(t) \rangle = R_k \sin^2 2\theta \sin^2 \left( \frac{\Delta\omega}{2} t \right) + \frac{1}{\mathcal{N}_k} \langle \alpha_{k,e}^r | N_{\alpha_\mu}^{k,r} | \alpha_{k,e}^r \rangle \quad (4.26)$$

$$\frac{1}{\mathcal{N}_k} \langle \alpha_{k,e}^r(t) | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r(t) \rangle + \frac{1}{\mathcal{N}_k} \langle \alpha_{k,e}^r(t) | N_{\alpha_\mu}^{k,r} | \alpha_{k,e}^r(t) \rangle - \frac{1}{\mathcal{N}_k} \langle \alpha_{k,e}^r | N_{\alpha_\mu}^{k,r} | \alpha_{k,e}^r \rangle = 1. \quad (4.27)$$

respectively, with  $R_k \equiv \frac{|U_k|^2}{\mathcal{N}_k} = \frac{1-|V_k|^2}{1-\sin^2 \theta |V_k|^2}$ .

In conclusion, eqs.(4.19) and (4.20) exhibit the corrections to the flavor oscillations coming from the condensate contributions.

The conventional (approximate) results (4.7) and (4.8) are obtained when the condensate contributions are missing (in the  $|V_k| \rightarrow 0$  limit).

Notice that the fraction of  $\alpha_e^{k,r}$  particles which is condensed into the vacuum  $|0\rangle_{1,2}$  is "frozen", i.e. does not oscillate in time, as it is easily seen by noting that  $e^{-iH_{1,2}t}|0\rangle_{1,2} = |0\rangle_{1,2}$  (cf. also eq. (4.15)).

It is remarkable that the corrections depend on the modulus  $k$  through  $|U_k|^2 = 1 - \frac{1}{2}Z_k$ . Since  $Z_k \rightarrow 0$  when  $k \rightarrow \infty$ , these corrections disappear in the infinite momentum limit. However, for finite  $k$ , the oscillation amplitude is depressed by a factor  $|U_k|^2$ : the depression factor ranges from 1 to  $\frac{1}{2}$  depending on  $k$  and on the masses values, according to the behaviour of the  $Z_k$  function. It is an interesting question to ask if an experimental test may show such a dependence of the flavor oscillation amplitude.

We stress that the limit  $k \rightarrow \infty$  of eqs.(4.19) and (4.20) gives an exact result and is not the result of the finite volume approximation.

Since the correction factor is related to the vacuum condensate, we see that the vacuum acts as a "momentum (or spectrum) analyzer" for the oscillating neutrinos: neutrinos with  $k \gg \sqrt{m_1 m_2}$  oscillate more than neutrinos with  $k \simeq \sqrt{m_1 m_2}$ , due to the vacuum structure. Such a vacuum spectral analysis effect may sum up to other effects (such as MSW effect [8] in the matter) in depressing or enhancing neutrino oscillations.

The above scheme is easily generalized to the oscillations in the matter. As well known [1,2,8], the two flavors oscillations picture is modified due to the different interaction of  $\nu_e$  and  $\nu_\mu$  with the electrons of the medium. There are two contributions to this interaction: the first one, coming from neutral current, amounts to  $-G_F n_n / \sqrt{2}$  and it is present for both  $\nu_e$  and  $\nu_\mu$ ; the second one, coming from charged current, is given by  $\sqrt{2}G_F n_e$  and it is present only for  $\nu_e$ . Here  $n_e$  and  $n_n$  are the electron and neutron densities, respectively. This produces a difference in the effective masses of  $\nu_e$  and  $\nu_\mu$ , which can be expressed in terms of new free fields  $\tilde{\nu}_1$  and  $\tilde{\nu}_2$  with new masses  $\tilde{m}_1$  and  $\tilde{m}_2$  and a new mixing angle  $\tilde{\theta}$ .



The mixing relations (2.1) are thus rewritten as:

$$\begin{aligned}\nu_e(x) &= \tilde{\nu}_1(x) \cos \tilde{\theta} + \tilde{\nu}_2(x) \sin \tilde{\theta} \\ \nu_\mu(x) &= -\tilde{\nu}_1(x) \sin \tilde{\theta} + \tilde{\nu}_2(x) \cos \tilde{\theta} .\end{aligned}\quad (4.28)$$

The tilde quantities are calculated by diagonalizing the hamiltonian of  $\nu_e$  and  $\nu_\mu$  in matter, which is:

$$H_{e,\mu}^{matter} = H_{e,\mu}^{vacuum} - G_F n_n / \sqrt{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sqrt{2} G_F n_e \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} . \quad (4.29)$$

The matter and the vacuum parameters are related as follows [1,2,8]:

$$\sin^2 2\tilde{\theta} \simeq \sin^2 2\theta \left( \frac{\Delta m^2}{\Delta \tilde{m}^2} \right) \quad (4.30)$$

$$\Delta \tilde{m}^2 = \left[ (D - \Delta m^2 \cos 2\theta)^2 + (\Delta m^2 \sin 2\theta)^2 \right]^{\frac{1}{2}} \quad (4.31)$$

and

$$\tilde{m}_{1,2}^2 = \frac{1}{2} (m_1^2 + m_2^2 + D \mp \Delta \tilde{m}^2) \quad (4.32)$$

where  $D = 2\sqrt{2}G_F n_e k$ .

When  $D = 2\sqrt{2}G_F n_e^{crit} k = \Delta m^2 \cos 2\theta$  there is resonance and  $\sin^2 2\tilde{\theta}$  goes to unity (MSW effect).

In our scheme it is possible to treat oscillations in matter starting with mixing relations (4.28) and repeating all the procedure described above. So we can use all the results obtained in our treatment simply by substituting  $\theta$ ,  $m_1$ ,  $m_2$  with the corresponding tilde quantities. In particular, the oscillation formula becomes in the matter:

$$\langle \alpha_{k,e}^r(t) | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r(t) \rangle = \left( 1 - \sin^2 \tilde{\theta} |\tilde{V}_k|^2 \right) - |\tilde{U}_k|^2 \sin^2 2\tilde{\theta} \sin^2 \left( \frac{\Delta \tilde{\omega}}{2} t \right) . \quad (4.33)$$

## 5 Three flavors fermion mixing

The extension of our discussion to three flavors is complicated by the proliferation of terms in the explicit computation of the quantities of interest. However, it is possible to extract some results from the structure of the annihilators, without explicitly writing the expression for the vacuum state.

Among the various possible parameterizations of the three fields mixing matrix, we choose to work with the following one:

$$M = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \quad (5.1)$$

with  $c_{ij} \equiv \cos \theta_{ij}$ ,  $s_{ij} \equiv \sin \theta_{ij}$ , since it is the familiar parameterization of CKM matrix [1].

To generate the M matrix, we define

$$G_{12}(\theta_{12}) = \exp(\theta_{12} L_{12}) \quad , \quad G_{23}(\theta_{23}) = \exp(\theta_{23} L_{23}) \quad , \quad G_{13}(\theta_{13}) = \exp(\theta_{13} L_{13}) \quad (5.2)$$

where

$$L_{12} \equiv \int d^3x \left( \nu_1^\dagger(x) \nu_2(x) - \nu_2^\dagger(x) \nu_1(x) \right) \quad (5.3a)$$

$$L_{23} \equiv \int d^3x \left( \nu_2^\dagger(x) \nu_3(x) - \nu_3^\dagger(x) \nu_2(x) \right) \quad (5.3b)$$

$$L_{13} \equiv \int d^3x \left( e^{i\delta} \nu_1^\dagger(x) \nu_3(x) - e^{-i\delta} \nu_3^\dagger(x) \nu_1(x) \right) \quad (5.3c)$$

so that

$$\nu_e^\alpha(x) = G_{12}^{-1} G_{13}^{-1} G_{23}^{-1} \nu_1^\alpha(x) G_{23} G_{13} G_{12} \quad (5.4a)$$

$$\nu_\mu^\alpha(x) = G_{12}^{-1} G_{13}^{-1} G_{23}^{-1} \nu_2^\alpha(x) G_{23} G_{13} G_{12} \quad (5.4b)$$

$$\nu_\tau^\alpha(x) = G_{12}^{-1} G_{13}^{-1} G_{23}^{-1} \nu_3^\alpha(x) G_{23} G_{13} G_{12} . \quad (5.4c)$$

The matrix M is indeed obtained by using the following relations:

$$[\nu_1^\alpha(x), L_{12}] = \nu_2^\alpha(x) \quad , \quad [\nu_1^\alpha(x), L_{23}] = 0 \quad , \quad [\nu_1^\alpha(x), L_{13}] = e^{i\delta} \nu_3^\alpha(x) \quad (5.5a)$$

$$[\nu_2^\alpha(x), L_{12}] = -\nu_1^\alpha(x) \quad , \quad [\nu_2^\alpha(x), L_{23}] = \nu_3^\alpha(x) \quad , \quad [\nu_2^\alpha(x), L_{13}] = 0 \quad (5.5b)$$

$$[\nu_3^\alpha(x), L_{12}] = 0 \quad , \quad [\nu_3^\alpha(x), L_{23}] = -\nu_2^\alpha(x) \quad , \quad [\nu_3^\alpha(x), L_{13}] = -e^{-i\delta} \nu_1^\alpha(x) . \quad (5.5c)$$

Notice that the phase  $\delta$  is unavoidable for three fields mixing, while it can be incorporated in the fields definition for two fields mixing.

The vacuum in the flavor representation is:

$$|0\rangle_{e\mu\tau} = G_{12}^{-1} G_{13}^{-1} G_{23}^{-1} |0\rangle_{123} . \quad (5.6)$$

We do not give here the explicit form of this state, which is very complicated and is a combination of all possible couples  $\alpha_{k,i}^{r\dagger} \beta_{-k,j}^{r\dagger}$  with  $i, j = 1, 2, 3$ . Nevertheless, we can obtain physical informations from the structure of the annihilators  $\alpha_{k,l}^r$ ,  $\beta_{k,l}^r$  ( $l = e, \mu, \tau$ ). In the reference frame  $k = (0, 0, |k|)$  we obtain (see Appendix E):

$$\alpha_{k,e}^r = c_{12} c_{13} \alpha_{k,1}^r + s_{12} c_{13} \left( U_{12}^{k*} \alpha_{k,2}^r + \epsilon^r V_{12}^k \beta_{-k,2}^{r\dagger} \right) + e^{i\delta} s_{13} \left( U_{13}^{k*} \alpha_{k,3}^r + \epsilon^r V_{13}^k \beta_{-k,3}^{r\dagger} \right) , \quad (5.7a)$$

$$\alpha_{k,\mu}^r = \left( c_{12} c_{23} - e^{-i\delta} s_{12} s_{23} s_{13} \right) \alpha_{k,2}^r - \left( s_{12} c_{23} + e^{-i\delta} c_{12} s_{23} s_{13} \right) \left( U_{12}^k \alpha_{k,1}^r - \epsilon^r V_{12}^k \beta_{-k,1}^{r\dagger} \right) +$$

$$+ s_{23}c_{13} \left( U_{23}^{k*} \alpha_{k,3}^r + \epsilon^r V_{23}^k \beta_{-k,3}^{r\dagger} \right) , \quad (5.7b)$$

$$\begin{aligned} \alpha_{k,\tau}^r = & c_{23}c_{13} \alpha_{k,3}^r - \left( c_{12}s_{23} + e^{-i\delta} s_{12}c_{23}s_{13} \right) \left( U_{23}^k \alpha_{k,2}^r - \epsilon^r V_{23}^k \beta_{-k,2}^{r\dagger} \right) + \\ & + \left( s_{12}s_{23} - e^{-i\delta} c_{12}c_{23}s_{13} \right) \left( U_{13}^k \alpha_{k,1}^r - \epsilon^r V_{13}^k \beta_{-k,1}^{r\dagger} \right) , \end{aligned} \quad (5.7c)$$

$$\beta_{-k,e}^r = c_{12}c_{13} \beta_{-k,1}^r + s_{12}c_{13} \left( U_{12}^{k*} \beta_{-k,2}^r - \epsilon^r V_{12}^k \alpha_{k,2}^{r\dagger} \right) + e^{-i\delta} s_{13} \left( U_{13}^{k*} \beta_{-k,3}^r - \epsilon^r V_{13}^k \alpha_{k,3}^{r\dagger} \right) , \quad (5.7d)$$

$$\begin{aligned} \beta_{-k,\mu}^r = & \left( c_{12}c_{23} - e^{i\delta} s_{12}s_{23}s_{13} \right) \beta_{-k,2}^r - \left( s_{12}c_{23} + e^{i\delta} c_{12}s_{23}s_{13} \right) \left( U_{12}^k \beta_{-k,1}^r + \epsilon^r V_{12}^k \alpha_{k,1}^{r\dagger} \right) + \\ & + s_{23}c_{13} \left( U_{23}^{k*} \beta_{-k,3}^r - \epsilon^r V_{23}^k \alpha_{k,3}^{r\dagger} \right) , \end{aligned} \quad (5.7e)$$

$$\begin{aligned} \beta_{-k,\tau}^r = & c_{23}c_{13} \beta_{-k,3}^r - \left( c_{12}s_{23} + e^{i\delta} s_{12}c_{23}s_{13} \right) \left( U_{23}^k \beta_{-k,2}^r + \epsilon^r V_{23}^k \alpha_{k,2}^{r\dagger} \right) + \\ & + \left( s_{12}s_{23} - e^{i\delta} c_{12}c_{23}s_{13} \right) \left( U_{13}^k \beta_{-k,1}^r + \epsilon^r V_{13}^k \alpha_{k,1}^{r\dagger} \right) . \end{aligned} \quad (5.7f)$$

From eqs.(5.7) we observe that, in contrast with the case of two flavors mixing, the condensation densities are now different for different flavors (cf. eq.(4.17)):

$${}_{123}\langle 0|N_{\alpha_e}^{k,r}|0\rangle_{123} = {}_{123}\langle 0|N_{\beta_e}^{k,r}|0\rangle_{123} = s_{12}^2 c_{13}^2 |V_{12}^k|^2 + s_{13}^2 |V_{13}^k|^2 , \quad (5.8a)$$

$$\begin{aligned}
& {}_{123}\langle 0|N_{\alpha_\mu}^{k,r}|0\rangle_{123} = {}_{123}\langle 0|N_{\beta_\mu}^{k,r}|0\rangle_{123} = \\
& = \left| s_{12}c_{23} + e^{-i\delta} c_{12}s_{23}s_{13} \right|^2 |V_{12}^k|^2 + s_{23}^2 c_{13}^2 |V_{23}^k|^2, \tag{5.8b}
\end{aligned}$$

$$\begin{aligned}
& {}_{123}\langle 0|N_{\alpha_\tau}^{k,r}|0\rangle_{123} = {}_{123}\langle 0|N_{\beta_\tau}^{k,r}|0\rangle_{123} = \\
& = \left| c_{12}s_{23} + e^{-i\delta} s_{12}c_{23}s_{13} \right|^2 |V_{23}^k|^2 + \left| s_{12}s_{23} - e^{-i\delta} c_{12}c_{23}s_{13} \right|^2 |V_{13}^k|^2. \tag{5.8c}
\end{aligned}$$

To study the three flavors neutrino oscillations, we observe that, in the finite volume approximation

$$|\alpha_{k,e}^r\rangle = c_{12}c_{13} |\alpha_{k,1}^r\rangle + s_{12}c_{13} |\alpha_{k,2}^r\rangle + e^{i\delta} s_{13} |\alpha_{k,3}^r\rangle. \tag{5.9}$$

The time evolution of this state is given by

$$|\alpha_{k,e}^r(t)\rangle = c_{12}c_{13} e^{-i\omega_1 t} |\alpha_{k,1}^r\rangle + s_{12}c_{13} e^{-i\omega_2 t} |\alpha_{k,2}^r\rangle + e^{i\delta} s_{13} e^{-i\omega_3 t} |\alpha_{k,3}^r\rangle. \tag{5.10}$$

The number operator is

$$\begin{aligned}
N_{\alpha_e}^{k,r} &= \alpha_{k,e}^{r\dagger} \alpha_{k,e}^r = \\
&= c_{12}^2 c_{13}^2 \alpha_{k,1}^{r\dagger} \alpha_{k,1}^r + c_{12} s_{12} c_{13}^2 \alpha_{k,1}^{r\dagger} \alpha_{k,2}^r + e^{i\delta} c_{12} c_{13} s_{13} \alpha_{k,1}^{r\dagger} \alpha_{k,3}^r + \\
&+ c_{12} s_{12} c_{13}^2 \alpha_{k,2}^{r\dagger} \alpha_{k,1}^r + s_{12}^2 c_{13}^2 \alpha_{k,2}^{r\dagger} \alpha_{k,2}^r + e^{i\delta} s_{12} c_{13} s_{13} \alpha_{k,2}^{r\dagger} \alpha_{k,3}^r + \\
&+ e^{-i\delta} c_{12} c_{13} s_{13} \alpha_{k,3}^{r\dagger} \alpha_{k,1}^r + e^{-i\delta} s_{12} c_{13} s_{13} \alpha_{k,3}^{r\dagger} \alpha_{k,2}^r + s_{13}^2 \alpha_{k,3}^{r\dagger} \alpha_{k,3}^r. \tag{5.11}
\end{aligned}$$

and thus

$$\begin{aligned}
\langle \alpha_{k,e}^r(t) | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r(t) \rangle &= c_{12}^4 c_{13}^4 + s_{12}^4 c_{13}^4 + s_{13}^4 + c_{12}^2 s_{12}^2 c_{13}^4 (e^{-i(\omega_2 - \omega_1)t} + e^{+i(\omega_2 - \omega_1)t}) + \\
&+ c_{12}^2 c_{13}^2 s_{13}^2 (e^{-i(\omega_3 - \omega_1)t} + e^{+i(\omega_3 - \omega_1)t}) + s_{12}^2 c_{13}^2 s_{13}^2 (e^{-i(\omega_3 - \omega_2)t} + e^{+i(\omega_3 - \omega_2)t})
\end{aligned}$$

i.e.

$$\begin{aligned}
\langle \alpha_{k,e}^r(t) | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r(t) \rangle &= 1 - \cos^4 \theta_{13} \sin^2 2\theta_{12} \sin^2 \left( \frac{\omega_2 - \omega_1}{2} t \right) + \\
&- \cos^2 \theta_{12} \sin^2 2\theta_{13} \sin^2 \left( \frac{\omega_3 - \omega_1}{2} t \right) - \sin^2 \theta_{12} \sin^2 2\theta_{13} \sin^2 \left( \frac{\omega_3 - \omega_2}{2} t \right). \tag{5.12}
\end{aligned}$$

On the other hand, the QFT computations give

$$\langle \alpha_{k,e}^r | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r \rangle = 1 - s_{12}^2 c_{13}^2 |V_{12}^k|^2 - s_{13}^2 |V_{13}^k|^2, \tag{5.13}$$

$${}_{123}\langle 0 | N_{\alpha_e}^{k,r} | 0 \rangle_{123} = s_{12}^2 c_{13}^2 |V_{12}^k|^2 + s_{13}^2 |V_{13}^k|^2, \tag{5.14}$$

and

$$\begin{aligned}
\langle \alpha_{k,e}^r(t) | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r(t) \rangle &= c_{12}^4 c_{13}^4 + |U_{12}^k|^2 s_{12}^4 c_{13}^4 + |U_{13}^k|^2 s_{13}^4 + |V_{12}^k|^2 c_{12}^2 s_{12}^2 c_{13}^4 + |V_{13}^k|^2 c_{12}^2 c_{13}^2 s_{13}^2 + \\
&+ \left( |U_{12}^k|^2 |V_{13}^k|^2 + |U_{13}^k|^2 |V_{12}^k|^2 \right) s_{12}^2 c_{13}^2 s_{13}^2 + |U_{12}^k|^2 c_{12}^2 s_{12}^2 c_{13}^4 (e^{-i(\omega_2 - \omega_1)t} + e^{+i(\omega_2 - \omega_1)t}) + \\
&+ |U_{13}^k|^2 c_{12}^2 c_{13}^2 s_{13}^2 (e^{-i(\omega_3 - \omega_1)t} + e^{+i(\omega_3 - \omega_1)t}) + |U_{12}^k|^2 |U_{13}^k|^2 s_{12}^2 c_{13}^2 s_{13}^2 (e^{-i(\omega_3 - \omega_2)t} + e^{+i(\omega_3 - \omega_2)t}),
\end{aligned}$$

namely

$$\begin{aligned}
\langle \alpha_{k,e}^r(t) | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r(t) \rangle &= \left( 1 - \sin^2 \theta_{12} \cos^2 \theta_{13} |V_{12}^k|^2 - \sin^2 \theta_{13} |V_{13}^k|^2 \right) + \\
&- |U_{12}^k|^2 \cos^4 \theta_{13} \sin^2 2\theta_{12} \sin^2 \left( \frac{\omega_2 - \omega_1}{2} t \right) - |U_{13}^k|^2 \cos^2 \theta_{12} \sin^2 2\theta_{13} \sin^2 \left( \frac{\omega_3 - \omega_1}{2} t \right) + \\
&- |U_{12}^k|^2 |U_{13}^k|^2 \sin^2 \theta_{12} \sin^2 2\theta_{13} \sin^2 \left( \frac{\omega_3 - \omega_2}{2} t \right), \tag{5.15}
\end{aligned}$$

which is to be compared with eq.(5.12).

## 6 Conclusions

In this paper we have studied the fermion mixing transformations in the QFT framework. In particular we have considered the Pontecorvo mixing transformations for neutrino Dirac fields [5]. In the LSZ formalism of QFT [3,6,7] the Fock space of definite flavor states is shown to be unitarily inequivalent in the infinite volume limit to the Fock space of definite mass states. The flavor states are obtained as condensate of massive neutrino-antineutrino pairs and exhibit the structure of  $SU(2)$  generalized coherent states [4]. The condensation density is computed as a function of the mixing angle, of the momentum modulus and of the neutrino masses.

The neutrino oscillation formula is derived and its amplitude turns out to be momentum dependent. We suggest that such a result may be object of experimental investigation.

In the  $k \rightarrow \infty$  limit the momentum dependence disappears and the oscillation formula reproduces the usual one. Notice however that the oscillation formula we obtain in the limit  $k \rightarrow \infty$  is exact and is not the result of the finite volume approximation used in the conventional treatment.

Since the oscillation term is related in our analysis to the vacuum condensate, the vacuum acts as a "momentum (or spectrum) analyzer" for the oscillating neutrinos. Such a vacuum spectral analysis effect may contribute in depressing or enhancing neutrino oscillations.

We observe that the functional dependence of the oscillating term on the momentum is such that, if experimentally tested, may give indication on the neutrino masses since the function  $Z_k = 2(1 - |U_k|^2)$  (cf. eqs.(3.10) and (4.19)) has a maximum at  $k = \sqrt{m_1 m_2}$ .

Although the physically relevant quantities are given by expectation values of the observables, nevertheless it is interesting to observe that the ratio of the amplitudes of the  $|\alpha_{k,1}^r\rangle$  and  $|\alpha_{k,2}^r\rangle$  components of the state  $|\alpha_{k,e}^r(t)\rangle$  is constant in time, as can be seen from eq.(4.14), and that such a feature persists even in the limit  $k \rightarrow \infty$  (i.e.  $|U_k| \rightarrow 1$ ) where, however, the oscillation formula (4.19) reduces to the usual one (4.7). This is in contrast with the finite volume approximation where "decoherence" between the components  $|\alpha_{k,1}^r\rangle$  and  $|\alpha_{k,2}^r\rangle$  arises from the phase factor  $\exp(-i\Delta\omega t)$  (see eq.(4.5)).

We have shown that our discussion can be extended to the oscillations in matter and the corresponding oscillation formula is obtained.

We have also studied the three flavors mixing and have obtained the corresponding oscillation formula, which also in this case is momentum dependent.

We stress the crucial rôle played by the existence in QFT of infinitely many unitarily inequivalent representations of the canonical anti-commutation rules. We have in fact explicitly shown that the neutrino mixing transformations map state spaces which are unitarily inequivalent in the infinite volume limit. In this way we realize the limit of validity of the identification of the vacuum state for definite mass neutrinos with the vacuum state for definite flavor neutrinos. We point out that such an identification is actually an approximation since the flavor vacuum has the structure of an  $SU(2)$  generalized coherent state and it is only allowed at finite volume. The vacua identification is meaningless in QFT since the mass and the flavor space are unitarily inequivalent.

Finally, we observe that although our discussion has been focused on the neutrino mixing, nevertheless it can be extended, with due changes, to other fermion mixing transformations, such as the CKM mixing transformations. In this last case, the so called free fields in the LSZ formalism are to be understood as the asymptotically free quark fields.

The study of the QFT for mixing of boson fields with different masses is also in progress. In that case preliminary results show [9] that relations analogous to eqs.(3.2) and (3.6) hold so that we have a non trivial vacuum structure. In the boson case we find  $|U_k| = \cosh \sigma_k$  and  $|V_k| = \sinh \sigma_k$ , with  $\sigma_k = \frac{1}{2} \log \left( \frac{\omega_{k,1}}{\omega_{k,2}} \right)$  where  $\omega_{k,i}$ ,  $i = 1, 2$ , is the boson energy.

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## Appendix A

Using the algebra (2.19) and the relations (2.27), we have:

$$\begin{aligned}
S_+^k S_-^k |0\rangle_{1,2} &= S_-^k S_+^k |0\rangle_{1,2} & (S_+^k)^2 (S_-^k)^2 |0\rangle_{1,2} &= (S_-^k)^2 (S_+^k)^2 |0\rangle_{1,2} \\
(S_+^k)^2 S_-^k |0\rangle_{1,2} &= S_-^k (S_+^k)^2 |0\rangle_{1,2} + 2S_+^k |0\rangle_{1,2} \\
(S_-^k)^2 S_+^k |0\rangle_{1,2} &= S_+^k (S_-^k)^2 |0\rangle_{1,2} + 2S_-^k |0\rangle_{1,2} \\
(S_+^k)^3 S_-^k |0\rangle_{1,2} &= 6(S_+^k)^2 |0\rangle_{1,2} & (S_+^k)^4 S_-^k |0\rangle_{1,2} &= 0 \\
(S_+^k)^3 (S_-^k)^2 |0\rangle_{1,2} &= 6S_-^k (S_+^k)^2 |0\rangle_{1,2} & (S_-^k)^3 (S_+^k)^2 |0\rangle_{1,2} &= 6S_+^k (S_-^k)^2 |0\rangle_{1,2} \\
(S_+^k)^4 (S_-^k)^2 |0\rangle_{1,2} &= 24(S_+^k)^2 |0\rangle_{1,2} & (S_+^k)^5 (S_-^k)^2 |0\rangle_{1,2} &= 0 \\
S_+^k S_-^k (S_+^k)^2 |0\rangle_{1,2} &= 4(S_+^k)^2 |0\rangle_{1,2} & S_-^k S_+^k (S_-^k)^2 |0\rangle_{1,2} &= 4(S_-^k)^2 |0\rangle_{1,2} \\
S_3^k S_-^k S_+^k |0\rangle_{1,2} &= 0 & S_3^k (S_+^k)^2 (S_-^k)^2 |0\rangle_{1,2} &= 0 \\
(S_3^k)^n S_-^k |0\rangle_{1,2} &= (-1)^n S_-^k |0\rangle_{1,2} & (S_3^k)^n (S_-^k)^2 |0\rangle_{1,2} &= (-2)^n (S_-^k)^2 |0\rangle_{1,2} \\
(S_3^k)^n S_+^k |0\rangle_{1,2} &= S_+^k |0\rangle_{1,2} & (S_3^k)^n (S_+^k)^2 |0\rangle_{1,2} &= 2^n (S_+^k)^2 |0\rangle_{1,2} \\
S_3^k S_-^k (S_+^k)^2 |0\rangle_{1,2} &= S_-^k (S_+^k)^2 |0\rangle_{1,2} & S_3^k S_+^k (S_-^k)^2 |0\rangle_{1,2} &= -S_+^k (S_-^k)^2 |0\rangle_{1,2}
\end{aligned}$$

Use of the above relations gives eq.(2.28).

## Appendix B

Wave functions and  $Z_k$ :

$$u_{k,i}^r(t) = \hat{u}_{k,i}^r e^{-i\omega_{k,i} t} = A_i \begin{pmatrix} \xi^r \\ \frac{\bar{\sigma} \cdot \bar{k}}{\omega_{k,i} + m_i} \xi^r \end{pmatrix} e^{-i\omega_{k,i} t} \quad , \quad v_{k,i}^r(t) = \hat{v}_{k,i}^r e^{i\omega_{k,i} t} = A_i \begin{pmatrix} \frac{\bar{\sigma} \cdot \bar{k}}{\omega_{k,i} + m_i} \xi^r \\ \xi^r \end{pmatrix} e^{i\omega_{k,i} t}$$

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \quad A_i \equiv \left( \frac{\omega_{k,i} + m_i}{2\omega_{k,i}} \right)^{\frac{1}{2}} \quad , \quad i = 1, 2 \quad , \quad r = 1, 2 \quad .$$

$$\hat{v}_{-k,1}^{1\dagger} \hat{u}_{k,2}^1 = -\hat{v}_{-k,1}^{2\dagger} \hat{u}_{k,2}^2 = A_1 A_2 \left( \frac{-k_3}{\omega_{k,1} + m_1} + \frac{k_3}{\omega_{k,2} + m_2} \right)$$

$$\hat{v}_{-k,1}^{1\dagger} \hat{u}_{k,2}^2 = \left( \hat{v}_{-k,1}^{2\dagger} \hat{u}_{k,2}^1 \right)^* = A_1 A_2 \left( \frac{-k_1 + ik_2}{\omega_{k,1} + m_1} + \frac{k_1 - ik_2}{\omega_{k,2} + m_2} \right)$$

Using the above relations eq.(2.32) is obtained.

Eq.(2.31) follows if one observe that

$$\begin{aligned} (S_-^k)^2 |0\rangle_{1,2} &= 2 \left[ \left( u_{k,2}^{1\dagger} v_{-k,1}^2 \right) \left( u_{k,2}^{2\dagger} v_{-k,1}^1 \right) - \left( u_{k,2}^{1\dagger} v_{-k,1}^1 \right) \left( u_{k,2}^{2\dagger} v_{-k,1}^2 \right) \right] \alpha_{k,2}^{1\dagger} \beta_{-k,1}^{2\dagger} \alpha_{k,2}^{2\dagger} \beta_{-k,1}^{1\dagger} |0\rangle_{1,2} = \\ &= Z_k e^{2i(\omega_{k,1} + \omega_{k,2})t} \alpha_{k,2}^{1\dagger} \beta_{-k,1}^{2\dagger} \alpha_{k,2}^{2\dagger} \beta_{-k,1}^{1\dagger} |0\rangle_{1,2} . \end{aligned}$$

## Appendix C

Relationships for the number operator and eq.(3.2):

$$[N_1^k, S_+^k] = \sum_{r,s} \left( (u_{k,1}^{r\dagger} u_{k,2}^s) \alpha_{k,1}^{r\dagger} \alpha_{k,2}^s + (u_{k,1}^{r\dagger} v_{-k,2}^s) \alpha_{k,1}^{r\dagger} \beta_{-k,2}^{s\dagger} \right)$$

$$[N_1^k, S_-^k] = - \sum_{r,s} \left( (u_{k,2}^{r\dagger} u_{k,1}^s) \alpha_{k,2}^{r\dagger} \alpha_{k,1}^s + (v_{-k,2}^{r\dagger} u_{k,1}^s) \beta_{-k,2}^{r\dagger} \alpha_{k,1}^s \right)$$

$$[N_1^k, S_-^k] |0\rangle_{1,2} = N_1^k S_-^k |0\rangle_{1,2} = N_1^k (S_-^k)^2 |0\rangle_{1,2} = 0$$

$$[N_1^k, S_-^k] S_-^k |0\rangle_{1,2} = [N_1^k, S_-^k] (S_-^k)^2 |0\rangle_{1,2} = [N_1^k, S_+^k] (S_+^k)^2 |0\rangle_{1,2} = 0$$

$$[N_1^k, S_+^k] |0\rangle_{1,2} = S_+^k |0\rangle_{1,2} \quad , \quad [N_1^k, S_+^k] S_+^k |0\rangle_{1,2} = (S_+^k)^2 |0\rangle_{1,2}$$

$$N_1^k S_+^k |0\rangle_{1,2} = S_+^k |0\rangle_{1,2} \quad , \quad N_1^k (S_+^k)^2 |0\rangle_{1,2} = 2(S_+^k)^2 |0\rangle_{1,2}$$

$${}_{1,2} \langle 0 | S_-^k N_1^k S_+^k | 0 \rangle_{1,2} = Z_k$$

$${}_{1,2} \langle 0 | S_-^k N_1^k S_-^k (S_+^k)^2 | 0 \rangle_{1,2} = {}_{1,2} \langle 0 | (S_-^k)^2 S_+^k N_1^k S_+^k | 0 \rangle_{1,2} = (Z_k)^2$$

$${}_{1,2} \langle 0 | S_+^k S_-^k N_1^k S_+^k S_-^k | 0 \rangle_{1,2} = Z_k + \frac{1}{2} (Z_k)^2$$

$${}_{1,2} \langle 0 | (S_-^k)^2 N_1^k (S_+^k)^2 | 0 \rangle_{1,2} = 2 (Z_k)^2$$

$${}_{1,2} \langle 0 | (S_-^k)^2 S_+^k N_1^k S_-^k (S_+^k)^2 | 0 \rangle_{1,2} = 6 (Z_k)^2$$

$${}_{1,2} \langle 0 | (S_+^k)^2 S_-^k N_1^k S_+^k (S_-^k)^2 | 0 \rangle_{1,2} = 2 (Z_k)^2$$

$${}_{1,2} \langle 0 | (S_+^k)^2 (S_-^k)^2 N_1^k S_+^k S_-^k | 0 \rangle_{1,2} = {}_{1,2} \langle 0 | S_+^k S_-^k N_1^k (S_+^k)^2 (S_-^k)^2 | 0 \rangle_{1,2} = 4 (Z_k)^2$$

$${}_{1,2} \langle 0 | (S_+^k)^2 (S_-^k)^2 N_1^k (S_+^k)^2 (S_-^k)^2 | 0 \rangle_{1,2} = 24 (Z_k)^2 .$$



Eq.(3.2) is then obtained as

$$\begin{aligned} {}_{e,\mu}\langle 0|N_{\alpha_1}^k|0\rangle_{e,\mu} &= Z_k(\sin^2\theta\cos^2\theta+\sin^4\theta)+ \\ &+(Z_k)^2\sin^4\theta(-\cos^2\theta+2\sin^2\theta\cos^2\theta-2\sin^2\theta+\frac{1}{2}\cos^4\theta+\frac{1}{2}+\frac{3}{2}\sin^4\theta)= \\ &= Z_k\sin^2\theta. \end{aligned}$$

## Appendix D

For the calculation of  $|0\rangle_{e,\mu}^k$  it is useful to choose  $k = (0, 0, |k|)$ . In this reference frame the operators  $S_+^k, S_-^k, S_3^k$  are written as follows:

$$\begin{aligned} S_+^k &\equiv \sum_{k,r} S_+^{k,r} = \\ &= \sum_r \left( U_k^* \alpha_{k,1}^{r\dagger} \alpha_{k,2}^r - \epsilon^r V_k^* \beta_{-k,1}^r \alpha_{k,2}^r + \epsilon^r V_k \alpha_{k,1}^{r\dagger} \beta_{-k,2}^{r\dagger} + U_k \beta_{-k,1}^r \beta_{-k,2}^{r\dagger} \right) \\ S_-^k &\equiv \sum_{k,r} S_-^{k,r} = \\ &= \sum_r \left( U_k \alpha_{k,2}^{r\dagger} \alpha_{k,1}^r + \epsilon^r V_k^* \beta_{-k,2}^r \alpha_{k,1}^r - \epsilon^r V_k \alpha_{k,2}^{r\dagger} \beta_{-k,1}^{r\dagger} + U_k^* \beta_{-k,2}^r \beta_{-k,1}^{r\dagger} \right) \\ S_3^k &\equiv \sum_{k,r} S_3^{k,r} = \frac{1}{2} \sum_{k,r} \left( \alpha_{k,1}^{r\dagger} \alpha_{k,1}^r - \beta_{-k,1}^{r\dagger} \beta_{-k,1}^r - \alpha_{k,2}^{r\dagger} \alpha_{k,2}^r + \beta_{-k,2}^{r\dagger} \beta_{-k,2}^r \right), \end{aligned}$$

where  $U_k, V_k$  have been defined in eqs.(3.7)-(3.10) and  $\epsilon^r = (-1)^r$ . It is easy to show that the  $su(2)$  algebra holds for  $S_{\pm}^{k,r}$  and  $S_3^{k,r}$ , which means that the  $su_k(2)$  algebra given in eqs.(2.19) splits into  $r$  disjoint  $su_{k,r}(2)$  algebras. Using the Gaussian decomposition,  $|0\rangle_{e,\mu}^k$  can be written as

$$|0\rangle_{e,\mu}^k = \prod_r \exp(-\tan\theta S_+^{k,r}) \exp(-2\ln\cos\theta S_3^{k,r}) \exp(\tan\theta S_-^{k,r}) |0\rangle_{1,2}$$

where  $0 \leq \theta < \frac{\pi}{2}$ . The final expression for  $|0\rangle_{e,\mu}^k$  in terms of  $S_{\pm}^{k,r}$  and  $S_3^{k,r}$  is then

$$|0\rangle_{e,\mu}^k = \prod_r \left[ 1 + \sin\theta\cos\theta \left( S_-^{k,r} - S_+^{k,r} \right) - \sin^2\theta S_+^{k,r} S_-^{k,r} \right] |0\rangle_{1,2},$$

from which we finally obtain eq.(3.11).

## Appendix E

Useful relations for three flavors mixing.

We work in the frame  $k = (0, 0, |k|)$  and for simplicity we omit the  $k$  and the helicity indices.

$$\left\{ \begin{array}{l}
G_{23}^{-1} \alpha_1 G_{23} = \alpha_1 \\
G_{13}^{-1} \alpha_1 G_{13} = c_{13} \alpha_1 + e^{i\delta} s_{13} (U_{13}^* \alpha_3 + \epsilon_r V_{13} \beta_3^\dagger) \\
G_{12}^{-1} \alpha_1 G_{12} = c_{12} \alpha_1 + s_{12} (U_{12}^* \alpha_2 + \epsilon_r V_{12} \beta_2^\dagger) \\
G_{23}^{-1} \alpha_2 G_{23} = c_{23} \alpha_2 + s_{23} (U_{23}^* \alpha_3 + \epsilon_r V_{23} \beta_3^\dagger) \\
G_{13}^{-1} \alpha_2 G_{13} = \alpha_2 \\
G_{12}^{-1} \alpha_2 G_{12} = c_{12} \alpha_2 - s_{12} (U_{12} \alpha_1 - \epsilon_r V_{12} \beta_1^\dagger) \\
G_{23}^{-1} \alpha_3 G_{23} = c_{23} \alpha_3 - s_{23} (U_{12} \alpha_2 - \epsilon_r V_{12} \beta_2^\dagger) \\
G_{13}^{-1} \alpha_3 G_{13} = c_{13} \alpha_3 - e^{-i\delta} s_{13} (U_{13} \alpha_1 - \epsilon_r V_{13} \beta_1^\dagger) \\
G_{12}^{-1} \alpha_3 G_{12} = \alpha_3 \\
G_{23}^{-1} \beta_1^\dagger G_{23} = \beta_1^\dagger \\
G_{13}^{-1} \beta_1^\dagger G_{13} = c_{13} \beta_1^\dagger + e^{i\delta} s_{13} (U_{13} \beta_3^\dagger - \epsilon_r V_{13}^* \alpha_3) \\
G_{12}^{-1} \beta_1^\dagger G_{12} = c_{12} \beta_1^\dagger + s_{12} (U_{12} \beta_2^\dagger - \epsilon_r V_{12}^* \alpha_2) \\
G_{23}^{-1} \beta_2^\dagger G_{23} = c_{23} \beta_2^\dagger + s_{23} (U_{23} \beta_3^\dagger - \epsilon_r V_{23}^* \alpha_3) \\
G_{13}^{-1} \beta_2^\dagger G_{13} = \beta_2^\dagger \\
G_{12}^{-1} \beta_2^\dagger G_{12} = c_{12} \beta_2^\dagger - s_{12} (U_{12}^* \beta_1^\dagger + \epsilon_r V_{12}^* \alpha_1) \\
G_{23}^{-1} \beta_3^\dagger G_{23} = c_{23} \beta_3^\dagger - s_{23} (U_{23}^* \beta_2^\dagger + \epsilon_r V_{23}^* \alpha_2) \\
G_{13}^{-1} \beta_3^\dagger G_{13} = c_{13} \beta_3^\dagger - e^{-i\delta} s_{13} (U_{13}^* \beta_1^\dagger + \epsilon_r V_{13}^* \alpha_1) \\
G_{12}^{-1} \beta_3^\dagger G_{12} = \beta_3^\dagger
\end{array} \right.$$

with

$$V_{ij}^k = |V_{ij}^k| e^{i(\omega_{k,j} + \omega_{k,i})t}, \quad U_{ij}^k = |U_{ij}^k| e^{i(\omega_{k,j} - \omega_{k,i})t}$$

$$|U_{ij}^k| = \left( \frac{\omega_{k,i} + m_i}{2\omega_{k,i}} \right)^{\frac{1}{2}} \left( \frac{\omega_{k,j} + m_j}{2\omega_{k,j}} \right)^{\frac{1}{2}} \left( 1 + \frac{k^2}{(\omega_{k,i} + m_i)(\omega_{k,j} + m_j)} \right)$$

$$|V_{ij}^k| = \left( \frac{\omega_{k,i} + m_i}{2\omega_{k,i}} \right)^{\frac{1}{2}} \left( \frac{\omega_{k,j} + m_j}{2\omega_{k,j}} \right)^{\frac{1}{2}} \left( \frac{k}{(\omega_{k,j} + m_j)} - \frac{k}{(\omega_{k,i} + m_i)} \right)$$

where  $i, j = 1, 2, 3$  and  $j > i$ , and

$$|U_{ij}^k|^2 + |V_{ij}^k|^2 = 1 \quad , \quad i = 1, 2, 3 \quad j > i$$

$$\left( V_{23}^k V_{13}^{k*} + U_{23}^{k*} U_{13}^k \right) = U_{12}^k \quad , \quad \left( V_{23}^k U_{13}^{k*} - U_{23}^{k*} V_{13}^k \right) = -V_{12}^k$$

$$\left( U_{12}^k U_{23}^k - V_{12}^{k*} V_{23}^k \right) = U_{13}^k \quad , \quad \left( U_{23}^k V_{12}^k + U_{12}^{k*} V_{23}^k \right) = V_{13}^k$$

$$\left( V_{12}^{k*} V_{13}^k + U_{12}^{k*} U_{13}^k \right) = U_{23}^k \quad , \quad \left( V_{12}^k U_{13}^k - U_{12}^k V_{13}^k \right) = -V_{23}^k .$$

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